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NUMERICAL VERIFICATION OF THE EXISTENCE OF THE ENERGY-CONCENTRATION EFFECT IN A HIGH-CONTRAST HEAVY-CHARGED COMPOSITE MATERIAL

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The existence of the energy-concentration effect in the regions between the inclusions of a high-contrast heavy-charged composite material has been verified in the process of numerical solution of the problem on heat conduction in this material. The formation of a high-energy region (energy "neck") under different conditions depending on the contrast range of the composite-material components and the distance between the inclusions has been investigated.

Introduction. The effect of energy concentration in the regions between the inclusions of a high-contrast heavy-charged composite material was predicted theoretically in [1, 2] and was used implicitly in many so-called "net" models defining high-contrast inhomogeneous media (see, e.g., [3]). As follows from the indicated works, this effect can be marked only in materials in which the distances between the inclusions are fairly small and, therefore, is difficult to investigate experimentally (the author failed to find a description of such experiments in the literature). The energy-concentration effect forms the basis for the so-called "net" models defining high-contrast heavy-charged composite materials (examples of the use of "net" models are presented, e.g., in [4]). Therefore, the verification of the existence of this effect is of practical and scientific interest. It should be noted that, in [1, 2], the existence of the energy-concentration effect was substantiated for the case of ideally conducting inclusions and for interinclusion distances close to zero. However, even through the conductivity of actual inclusions is high, it has a definite value, and the small distances between the inclusions are also of definite value. It is the author's opinion that the consideration of the influence of the finiteness of these quantities on the energy-concentration effect in the regions between the inclusions is also of theoretical and practical importance. This problem is difficult to solve analytically; therefore, we solved it with the use of a numerical method, namely, the finite-element method.

Periodic Problem. Numerical Verification of the Existence of the Energy-Concentration Effect. Let us consider a periodic system of nonintersecting disks that do not touch one another (Fig. 1). The regions $\{D_i, i = 1, ..., N\}$ will serve as inclusions, and the region $Q = R^2 \setminus \bigcup_{i=1}^{N} D_i$ will be a matrix.

The equation of heat conduction [5] at a temperature $\Theta(\mathbf{x})$ has the form

$$\operatorname{div} (a (\mathbf{x}) \nabla\Theta) = 0 \quad \text{in} \quad R^2 . \tag{1}$$

The heat conductivity coefficient is equal to

$$a (\mathbf{x}) = \begin{cases} a_{\rm m} & \text{in the matrix ,} \\ a_{\rm d} & \text{in the disks .} \end{cases} \tag{2}$$

The problem is characterized by two parameters — the contrast range (contrast) of the components of a composite material and the characteristic relative distance between the inclusions

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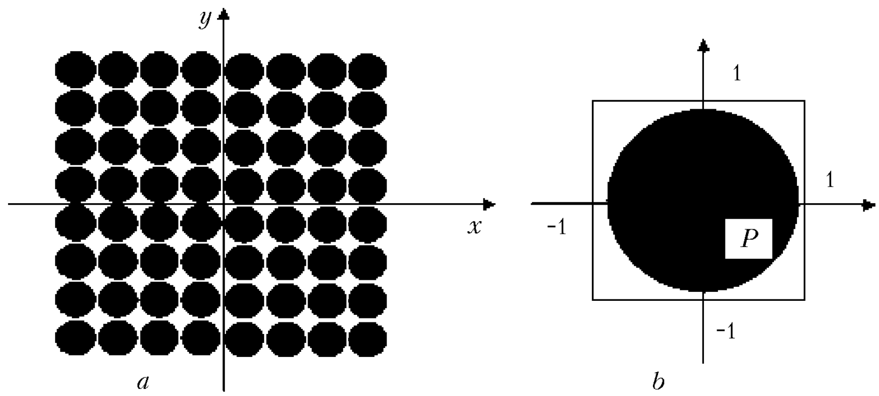

Fig. 1. Periodic system of disks (a) and a periodicity cell P (b).

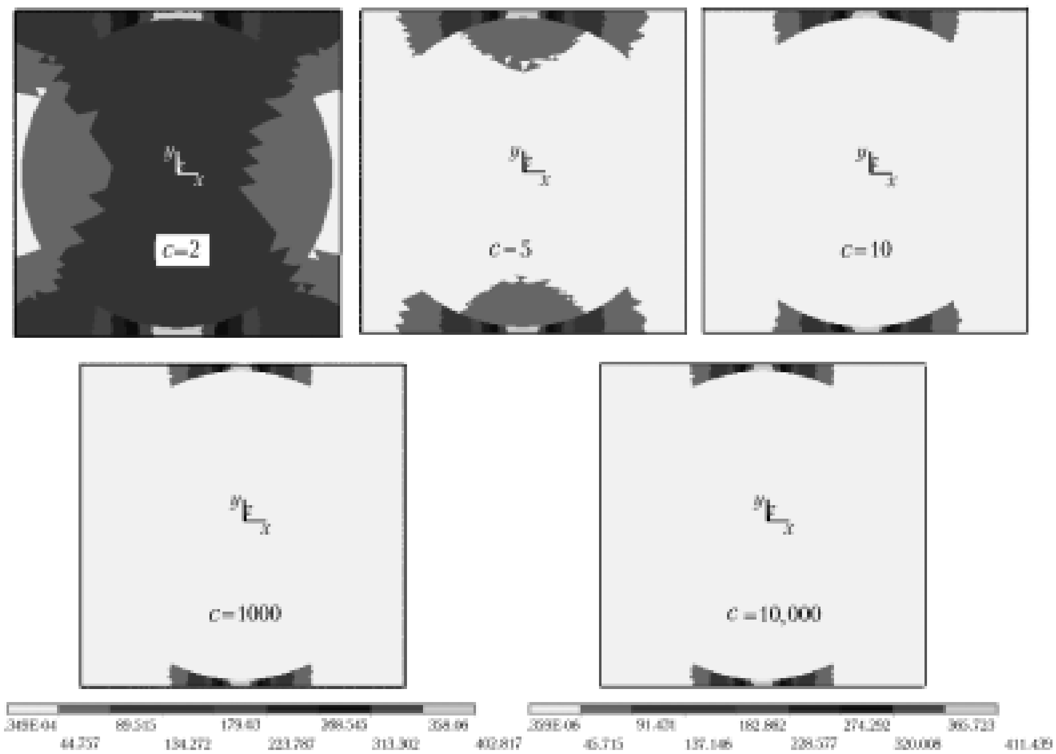

Fig. 2. Distribution of the energy density in the periodicity cell P (δ = 0.1, c = 0.05).

$$c = \frac{a_{\rm d}}{a_{\rm m}}, \quad \delta = \frac{d}{D}. \tag{3}$$

For a high-contrast heavy-charged composite material

$$c >> 1, \quad \delta << 1. \tag{4}$$

A heavy-charged composite material contains a maximum possible number of inclusions. The high charging of this material is equivalent to a dense packing of inclusions.

Below we consider a composite material with a periodic structure. If an infinite composite material (Fig. 1) has a thermal field with an average temperature gradient G directed along the $0y$ axis, the heat-conduction problem (1) for it is solved as

$$\Theta(\mathbf{x}) = Gy + \Theta_P(\mathbf{x}) , \tag{5}$$

where $\Theta_P(\mathbf{x})$ is a periodic function with a periodicity cell P (Fig. 1).

Representation (5) allows one to pass from the solution of problem (1) to the solution of the next problem for the periodicity cell:

$$\operatorname{div}(a(\mathbf{x})\nabla\Theta) = 0 \quad \text{in} \quad P , \quad \Theta(x, 0) = -1 , \quad \Theta(x, 1) = 1 ,$$

$$\frac{\partial\Theta}{\partial\mathbf{n}}(0, y) = \frac{\partial\Theta}{\partial\mathbf{n}}(1, y) = 0 . \tag{6}$$

Problem (6) was solved numerically for values of the contrast c falling within the range 2–10,000 and the characteristic relative interinclusion distances δ ranging from 0.05 to 0.005. It was assumed in the numerical calculations that the conductivity of the matrix is equal to unity and the radius of the disk is equal to unity (the disk diameter $D = 2$). A typical distribution of the doubled energy density $E = a(\mathbf{x})|\nabla\Theta|^2$ is shown in Fig. 2. It is seen that the energy neck begins to form geometrically when the contrast is equal to 5 (see Fig. 2). However, at small values of the contrast c, the energy density of the neck depends substantially on the value of c, and the energy density of the neck is stabilized at large values of the contrast (see the two lower diagrams in Fig. 2 and the corresponding scales of values under them). It was established that the energy density of the neck is stabilized at $c = 1000$. The author believes that this energy-density stabilization corresponds to the effect of energy concentration in the regions between the inclusions of a high-contrast heavy-charged composite material, described in [1, 2].

Nonperiodic Problem. Numerical Investigation of the Approximation of a Continuous Problem with the Use of a Net Model. The solution of the periodic problem on heat conduction in a high-contrast heavy-charged composite material allows one to determine the contrast range and the distance between the inclusions, at which there arises the energy-concentration effect in the regions between the inclusions of this material. We will solve the general (nonperiodic) heat-conduction problem for this material.

As was noted above, the energy-concentration effect forms the basis for the so-called "net" models defining high-contrast heavy-charged composite materials. Let us consider the problem on the accuracy of approximation of the continuous heat-conduction problem

$$\operatorname{div}(a(\mathbf{x})\nabla\Theta) = 0 \quad \text{in the region} \quad [-1, 1]^2 , \ \Theta(x, 0) = -1 , \ \Theta(x, 1) = 1 ,$$

$$\frac{\partial\Theta}{\partial\mathbf{n}}(0, y) = \frac{\partial\Theta}{\partial\mathbf{n}}(1, y) = 0 \tag{7}$$

by a net model. Problem (7) defines the temperature distribution in a square, the lateral faces of which are heat-insulated and the temperatures of the upper and lower faces are equal to 1 and −1. The net model includes the equations [1, 2, 4]

$$\sum_{j \cup N_i} C_{1j}^{(2)}(t_i - t_j) = 0 \tag{8}$$

at the nodes corresponding to the inclusions located inside the region Q; in this case,

$$t_i = -1 , \quad t_i = 1 \tag{9}$$

at the nodes corresponding to the lower and upper boundaries of the region $[-1, 1]^2$.

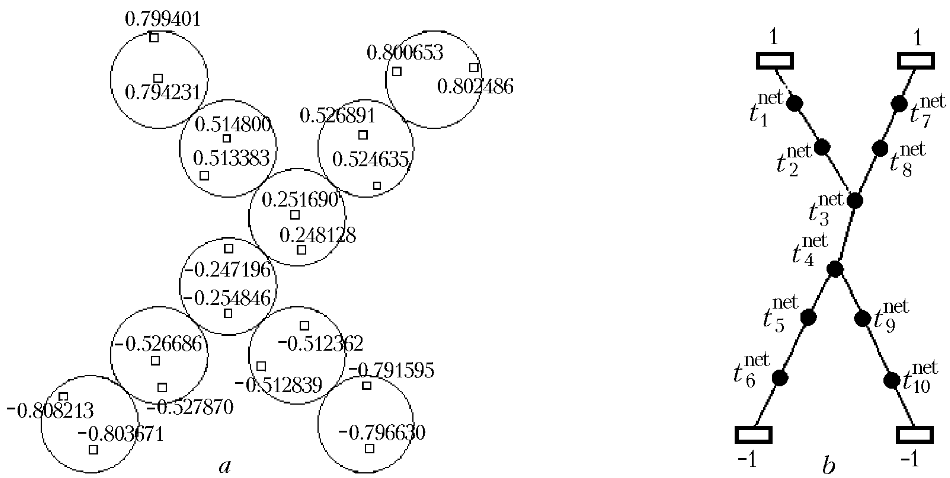

Fig. 3. System of disks and potentials determined from the solution of the continuous problem (a) and net model of the system of disks (b) (circles are net nodes corresponding to disks, and rectangles are nodes at the boundary of the region).

The capacitance of the disk–disk pair is equal to $\pi\sqrt{\frac{R}{\delta}}$ and the capacitance of the disk–semiplane pair is equal to $\pi\sqrt{\frac{2R}{\delta}}$, where δ is the distance between the disks or between the disk and the semiplane [6].

The continuous model of the composite material with a heat conduction defined by problem (7) is presented in Fig. 3a, and its net model (8), (9) is presented in Fig. 3b. As the criterion of approximation of problem (7) by model problem (8), (9), we used the difference between the temperatures of the disks and the temperatures at the nodes of the net model. In this case, the temperatures of the disks were determined from the numerical solution of problem (7) by the finite-element method, and the temperatures at the nodes of the net were determined from the solution of the system of linear algebraic equations. The configuration of the disks, for which the problem was solved, is presented in Fig. 3. The contrast range was assumed to be equal to 10^3–10^6 and the distances between the disks were taken from the interval 0.05–0.0005. Figure 3a shows the temperatures at certain points of the disks, determined for a contrast $c = 1000$ and a relative distance between the disks $\delta \approx 0.005$. The number near a small square is the temperature at its center. It is seen that the temperature inside each disk remains practically constant. This means that the approximation was made correctly. It should be noted that the constancy of the temperature inside the disks is consistent with the corresponding hypothesis of the net model [1–3].

According to [1, 2], the main hypothesis of the net model is the existence of the energy-concentration effect in the regions between the inclusions of a high-contrast heavy-charged composite material. For the purpose of verification of this fact, the data on the energy-density distribution were displayed on a computer. In the figure obtained, the region outside the disks with energy density close to zero and the small regions of the necks between the disks with nonzero (fairly large) energy densities are marked in different colors for clearness.

Figure 4a shows the distribution of the energy density in the neck between disks No. 3 and No. 4 in the black-white variant (the numbering of disks in Fig. 3b).

In Fig. 4b, the finite-element net in the channel between the disks and in the disks is shown. The black color of the channel corresponds to a large number of finite elements (about 4000). The use of this fine net in the neck is explained by the fact that energy is concentrated in the regions between the inclusions with the use of coarse nets, and we obtained solutions whose the graphical representation showed that the thermal field is distributed nonuniformly in the form of corners, deflections, etc., not characteristic of the solution of the heat-conduction equation.

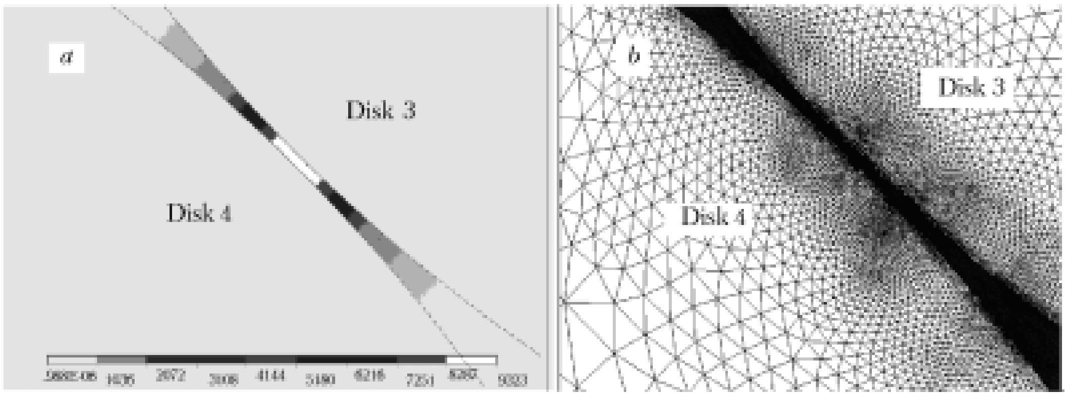

Fig. 4. Distribution of the energy density in the channel between the disks (a) and finite-element net in the channel between the disks and in the disks (b).

TABLE 1. Values of t_i^{cont} and t_i^{net} Determined from the Solution of Problems (7) and (8), (9) at Different Disk–Disk and Disk–Boundary Distances

i	$\delta = 0.005,\ \delta_b = 0.005$			$\delta = 0.0027,\ \delta_b = 0.0025$			$\delta = 0.00049,\ \delta_b = 0.0005$	
	t_i^{net}	t_i^{cont}		t_i^{net}	t_i^{cont}		t_i^{net}	$t_i^{\text{cont}},\ c = 10^4$
		$c = 10^3$	$c = 10^6$		$c = 10^3$	$c = 10^6$		
1	0.81	0.80	0.79	0.80	0.80	0.80	0.80	0.81
2	0.54	0.51	0.51	0.52	0.54	0.52	0.53	0.54
3	0.27	0.25	0.25	0.25	0.27	0.26	0.26	0.27
4	–0.27	–0.25	–0.25	–0.26	–0.27	–0.26	–0.26	–0.27
5	–0.54	–0.53	–0.52	–0.53	–0.54	–0.53	–0.53	–0.54
6	–0.81	–0.80	–0.80	–0.81	–0.80	–0.81	–0.81	–0.81
7	0.81	0.80	0.80	0.81	0.80	0.81	0.80	0.81
8	0.54	0.52	0.52	0.53	0.54	0.53	0.53	0.54
9	–0.54	–0.51	–0.51	–0.52	–0.54	–0.52	–0.53	–0.54
10	–0.81	–0.79	–0.80	–0.80	–0.80	–0.80	–0.80	–0.81

Let us denote the distances between the neighboring disks by δ and the distance between the near-boundary disks and the boundary by δ_b. The temperatures of the disks, determined from the solution of the continuous problem (7), will be designated as t_i^{cont}, where i is a number of a disk (as was noted above, the temperature inside a disk is practically constant and can be defined by one number). The temperatures of the nodes, determined from the solution of the net problem (8), (9), will be designated as t_i^{net}, where i is a number of a node. Table 1 presents the values of t_i^{cont} and t_i^{net}, determined from the solution of problems (7) and (8), (9), at different values of the distances δ, δ_b and the contrast c (3); the numbers of nodes in the table are identical to those in Fig. 3b. The relative error is determined as

$$\varepsilon_r = \max_{i=1,\dots,N} \left| \frac{t_i^{\text{cont}} - t_i^{\text{net}}}{t_i^{\text{cont}}} \right| \tag{10}$$

and the absolute error is equal to

$$\varepsilon_a = \max_{i=1,\dots,N} \left| t_i^{\text{cont}} - t_i^{\text{net}} \right| . \tag{11}$$

TABLE 2. Errors in the Solution of Problem (7) and (8), (9) (see Table 1)

δ	δ_b	ε_r	ε_a
0.005	0.005	0.080	0.03
0.0027	0.0025	0.074	0.02
0.00049	0.0005	0.037	0.01

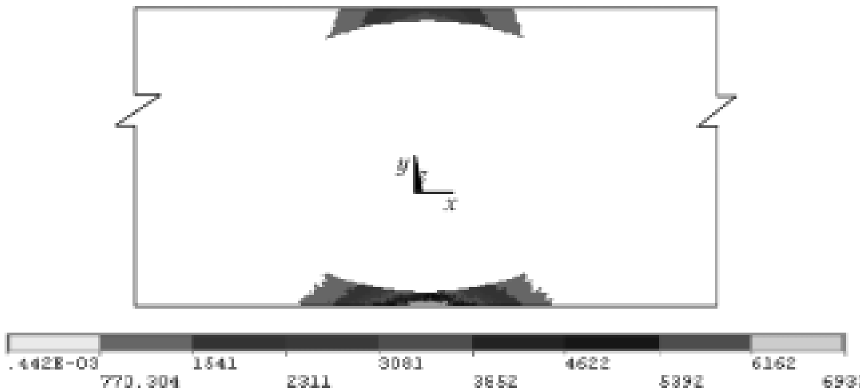

Fig. 5. Distribution of the energy density in the nonlinear composite material (δ = 0.05).

The values of these errors are given in Table 2, from which it is seen that, as the distances between the neighboring disks and the near-boundary disks and the boundaries decrease, the corresponding errors decrease, i.e., the solutions of problem (8), (9) approach the solutions of problem (7) in the disks. This allows the conclusion that the solutions of the grid problem (8), (9) approximate the solutions of problem (7) in the disks.

It should be noted that the solutions of problem (8), (9) approach the solutions of problem (7) in the disks fairly slowly with decrease in the distances δ and δ_b. The absolute error in our numerical calculations was 0.05 for the distance 0.015, 0.03 for the distance 0.005, 0.02 for the distance 0.0027, and 0.01 for the distance 0.00049. In this case, it was assumed that the temperature at the lower boundary is equal to -1, and the temperature at the upper boundary is equal to 1 (see Fig. 3). Accordingly, the temperature inside the region between the inclusions was determined within the same limits. The distances 0.015–0.01 can be realized in the case where the inclusions are closely packed, and the distances 0.005–0.001 are difficult to realize in practice; they are of interest mainly for mathematical investigation of the approximation of the solutions of problem (7) by problem (8), (9).

The data presented allow the conclusion that the solutions of problem (8), (9) approximate the solutions of problem (7) in the disks at a distance of 0.015 or smaller between them and a contrast of 103 or larger. At distances attainable in practice (0.015–0.01), the absolute approximation error is of the order of 0.03 and the relative approximation error is of the order of 0.08.

Existence of the Energy-Concentration Effect in a Composite Material with a Nonlinear Matrix. The problem on the existence of the energy-concentration effect in the regions between the inclusions of a nonlinear high-contrast heavy-charged composite material was numerically investigated. The problem was solved using the heat-conduction equation with a temperature-dependent heat-conductivity coefficient

$$\mathrm{div}\,(a\,(\mathbf{x}, \Theta)\,\nabla\Theta) = 0\,. \qquad (12)$$

The cases of existence of one inclusion and a system of inclusions, such as the system presented in Fig. 3a, were considered.

In the numerical calculations, the radius of a disc was taken to be equal to unity. It was assumed that the material being investigated has the following nonlinearity:

$$a(\mathbf{x}, \Theta) = \begin{cases} 1 + \frac{19}{4}(\Theta + 2) & \text{in the matrix ,} \\ 10^4 & \text{inside inclusions} \end{cases}$$

(the matrix in nonlinear and the inclusions are linear). In the problem being considered, it was assumed that $-1 \le \Theta \le 1$.

In both cases (the problem for one inclusion and the problem for a system of inclusions), the energy necks were clearly seen. For the first problem, the energy-density distribution in the nonlinear formulation is presented in Fig. 5. The characteristic energy values are indicated on the scale in Fig. 5.

It should be noted that the energy neck in the nonlinear problem differs somewhat from the energy necks in the linear problem (see Fig. 2), which is evidently explained by the nonlinearity of the first-mentioned problem. The author restricts himself to the determination of the existence of the energy-concentration effect in the regions between the inclusions of a nonlinear high-contrast heavy-charged composite material. A more comprehensive analysis of nonlinear problems and corresponding net models calls for a separate investigation.

Remark on the Choice of a Net for Numerical Solution of Problems for High-Contrast Heavy-Charged Composite Materials. In a number of works, the problems on heat conduction in composite materials were solved by the finite-element method with the use of a uniform net (see, e.g., [7, 8]). The possibility of use of uniform nets for calculating the heat conductivity of high-contrast heavy-charged composite materials was investigated in [1], where it was shown that, in the regions located at large distances from the "necks," the energy concentration is small and a uniform division of these regions does not prove its value (leads to an idle counting). Note that a uniform division is not warranted in a neck too, which, however, is explained by a different reason. In the above-mentioned investigations [7, 8], a uniform division resulted in that the region between the inclusions contained several (3–5) finite elements. An analysis performed by the author of the present work has shown that the temperature field in the region between the neighboring inclusions cannot be correctly calculated with this number of finite elements. The minimum number of finite elements, necessary for a correct calculation, should be several tens or, for more exact calculations, several hundreds. Consequently, to numerically solve the problem on heat conduction in a high-contrast heavy-charged composite material by the finite-element method, it is necessary to use a very nonuniform net accounting for the effect of energy concentration in the regions between the inclusions of this material, i.e., a finite-element net should be constructed with account for the microgeometry of the composite material.

Conclusions. The results of the numerical solution of the heat-conduction problem for a high-contrast heavy-charged composite material lend credence to the assumption that, in this material there exist energy "necks" and the energy-concentration effect can arise in the regions between the inclusions. The energy "necks" manifest themself markedly at a contrast $c \ge 10$ and a relative distance between the neighboring inclusions $\delta \le 0.1$. However, the energy density in a "neck" is stabilized at a contrast $c \ge 1000$ and the continuous problem can be approximated by the net problem with an accuracy of 10% (which can be considered as an acceptable accuracy for the problems on the study of materials [9]) beginning with the relative distance between the neighboring inclusions $\delta \le 0.015$.

The results of our numerical calculations have shown that the energy-concentration effect arises in the regions between the inclusions of a nonlinear heavy-charged composite material, even though the structure of the field in a "neck" is not entirely identical to that in the linear case.

The results of the work were reported in [10].

NOTATION

$a(\mathbf{x})$, heat-conductivity coefficient of a composite material as a function of the spatial variable $\mathbf{x}$; a_{m}, heat-conductivity coefficient of a matrix material; a_{d}, heat-conductivity coefficient of the inclusion material; $c = a_{\mathrm{d}}/a_{\mathrm{m}}$, contrast of the composite-material components; $C_{ij}^{(2)}$, capacitance of the ith and jth bodies everywhere over the space between the inclusions; d, characteristic distance between the inclusions; D, characteristic size (diameter) of the inclusions; D_i, system of disks; N, number of disks; $\mathbf{n}$, vector of the normal to the boundary of the region between the inclusions; Q, region occupied by the matrix; $R = D/2$, radius of a disk; t_i, temperature at the nodes of a net; t_i^{cont}, temperature of the disks determined from the solution of the continuous problem (7); t_i^{net}, temperature at the nodes determined from the solution of the net problem (8), (9); $\delta = d/D$, relative distance between the inclusions; δ_b, distance between the near-boundary disk and the boundary; ε_r, relative error of solution; ε_a, absolute error of solution; $\Theta(\mathbf{x})$, temperature as a function of the spatial variable **x**. Subscripts: a, absolute; b, boundary; cont, continuous; d, disk; m, matrix; net, net; r, relative.

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